

An Algebraic Perspective on Multivariate Tight Wavelet Frames. II

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March 11, 2014

Abstract

Continuing our recent work in [5] we study polynomial masks of multivariate tight wavelet frames from two additional and complementary points of view: convexity and system theory. We consider such polynomial masks that are derived by means of the unitary extension principle from a single polynomial. We show that the set of such polynomials is convex and reveal its extremal points as polynomials that satisfy the quadrature mirror filter condition. Multiplicative structure of such polynomial sets allows us to improve the known upper bounds on the number of frame generators derived from box splines. In the univariate and bivariate settings, the polynomial masks of a tight wavelet frame can be interpreted as the transfer function of a conservative multivariate linear system. Recent advances in system theory enable us to develop a more effective method for tight frame constructions. Employing an example by S. W. Drury, we show that for dimension greater than 2 such transfer function representations of the corresponding polynomial masks do not always exist. However, for wavelet masks derived from multivariate polynomials with non-negative coefficients, we determine explicit transfer function representations. We illustrate our results with several examples.

Keywords: multivariate wavelet frame, positive polynomial, sum of hermitian squares, transfer function.

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Math. Sci. Classification 2000: 65T60, 14P99 11E25, 90C26, 90C22.

1 Introduction

A tight wavelet frame of $L_2(\mathbb{R}^d)$ is determined, via Fourier transform, by a finite set of trigonometric polynomials p, a_1, \dots, a_N . The trigonometric polynomial p enters as the unique ingredient into the multiplicative identity

$$\hat{\phi}(M^T \theta) = p(z) \hat{\phi}(\theta), \quad \theta \in \mathbb{R}^d, \quad z_j = e^{i\theta_j}, \quad (1)$$

where M is a $d \times d$ matrix with integer entries whose eigenvalues are greater than 1 in absolute value. The identity (1) is called the two-scale relation, as it defines a representation of ϕ in terms of shifts of scaled versions of ϕ , i.e.

$$\phi(x) = |\det M| \sum_{\alpha \in \mathbb{Z}^d} p(\alpha) \phi(Mx - \alpha), \quad x \in \mathbb{R}^d. \quad (2)$$

Here, $p(z) = \sum_{\alpha \in \mathbb{Z}^d} p(\alpha) z^\alpha$ has finitely many nonzero coefficients $p(\alpha)$ and $z^\alpha = z_1^{\alpha_1} \dots z_d^{\alpha_d}$.

The translation group $G = 2\pi M^{-T} \mathbb{Z}^d / 2\pi \mathbb{Z}^d$ plays a central role in the discussion of the two-scale relation. Clearly, G is a finite group of order $m = |\det M|$. Throughout this article we maintain the notation and terminology introduced in [5]. Our main object of study, as in the previous article [5], is the *mask* p , regarded as a Laurent polynomial or, equivalently, a trigonometric polynomial on the d -dimensional torus

$$\mathbb{T}^d = \{z = (z_1, \dots, z_d) \in \mathbb{C}^d : |z_j| = 1 \text{ for } j = 1, \dots, d\}.$$

An element of the group $\sigma = (\sigma_1, \dots, \sigma_d) \in G$ acts on $p \in \mathbb{C}[\mathbb{T}^d]$ by

$$p^\sigma(z) := p(e^{-i\sigma_1} z_1, \dots, e^{-i\sigma_d} z_d), \quad z \in \mathbb{T}^d.$$

The conditions

$$p^\sigma(1, 1, \dots, 1) = \delta_{0, \sigma}, \quad \sigma \in G, \quad (3)$$

are called zero conditions or sum rules of order 1 in the literature, see [17] and references therein, and are important for the analysis of various properties of ϕ . Another important ingredient of the analysis is the fact that the support of ϕ is contained in the convex hull of $\{\alpha \in \mathbb{Z}^d : p(\alpha) \neq 0\}$.

We let $F_p = (p^\sigma)_{\sigma \in G}$, $F_{a_j} = (a_j^\sigma)_{\sigma \in G} : \mathbb{T}^d \rightarrow \mathbb{C}^m$ be column vectors. Then the identity

$$I_m - F_p(z) F_p(z)^* = \sum_{j=1}^N F_{a_j}(z) F_{a_j}(z)^* \quad (4)$$

is called the *Unitary Extension Principle* (UEP) in the seminal work on frames and shift-invariant spaces by Ron and Shen [23]. Here, $F_p(z)^* = \overline{F_p(z)}^T$ denotes complex conjugation and transposition. If the identities (3) and (4) are satisfied, then the functions

$$\psi_j(x) = |\det M| \sum_{\alpha \in \mathbb{Z}^d} a_j(\alpha) \phi(Mx - \alpha), \quad x \in \mathbb{R}^d, \quad (5)$$

are the generators of a tight wavelet frame; i.e. the family

$$X(\Psi) = \{m^{j/2} \psi_l(M^j \cdot -k) : 1 \leq l \leq N, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$$

defines a tight frame of $L_2(\mathbb{R}^d)$. Therefore, the UEP is the core of many constructions of tight wavelet frames, see e.g. [6, 7, 9, 11, 14, 22, 23, 26].

The constraint

$$f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^{\sigma} \geq 0 \quad (6)$$

is known in the literature as the sub-QMF condition on the trigonometric polynomial $p \in \mathbb{C}[\mathbb{T}^d]$. Due to $f = \det(I_m - F_p F_p^*)$, the condition in (6) is necessary for the existence of a_1, \dots, a_N that satisfy the UEP identities in (4).

In the first part of this article we investigate the convex structure of the set of trigonometric polynomials p subject to the restrictions (3) and (6). The following certificate of the positivity condition turns out to be of great importance

$$f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^{\sigma} = \sum_{j=1}^L |h_j|^2 \quad (7)$$

where h_j are G -invariant trigonometric polynomials. This certificate is called *sum of hermitian squares* (sos) decomposition of the non-negative trigonometric polynomial f . It was shown in [5] and [22] that the condition (7) is necessary and sufficient for the existence of trigonometric polynomials a_j in (4). The convex structure of the trigonometric polynomials p subject to the restrictions (3) and (7) is more complicated, as this set is not closed. However, we prove that the extremal points of the underlying convex sets coincide. We also show that the sets of those p satisfying either (6) or (7) are closed under multiplication. Consequently, we investigate the number of squares L in (7) of a product $p = p_1 p_2$ in terms of those of the factors p_1 and p_2 . Combined with the construction of tight wavelet frames in [22], we obtain better bounds for the number of tight frame generators for a large

class of trigonometric polynomials p , including the masks of multivariate box splines.

So far, we treated the mask p and $f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^{\sigma}$ as trigonometric polynomials. Since our analysis is not affected by multiplication of p by a fixed monomial $e^{i\beta\theta}$, we can assume that $p(z) = \sum_{\alpha \in \mathbb{N}_0^d} p(\alpha) z^{\alpha}$ is a polynomial in $z_j = e^{i\theta_j}$. Therefore, in Section 4, we consider p as a complex analytic polynomial $p \in \mathbb{C}[z]$ and rephrase the decomposition (7) as

$$f(z, \bar{z}) = 1 - \sum_{\sigma \in G} p^{\sigma}(z)^* p^{\sigma}(z) = \sum_{j=1}^L h_j(z)^* h_j(z) + R(z, \bar{z})$$

where p, h_j are complex analytic polynomials, z belongs to the polydisk

$$\mathbb{D}^d = \{z = (z_1, \dots, z_d) : |z_j| < 1, j = 1, \dots, d\},$$

and the residual part $R(z, \bar{z})$ vanishes on the torus.

This slight change of perspective brings into focus the complex analytic, vector valued polynomial

$$F_p(z) = (p^{\sigma}(z))_{\sigma \in G} \in \mathbb{C}^d, \quad z \in \mathbb{D}^d,$$

subject, by the maximum principle, to the contractivity condition

$$F_p(z)^* F_p(z) \leq 1, \quad z \in \mathbb{D}^d.$$

Analytic functions as above, from the polydisk to the unit ball $F_p : \mathbb{D}^d \rightarrow B(0, 1) \subseteq \mathbb{C}^m$, were intensively studied for more than a century. The classical works of Schur, Carathéodory, Fejér and Nevanlinna have completely settled the intricate structure of analytic functions from the disk to the disk or the half-plane. The rather independent and self-sustaining field of bounded analytic interpolation in one or several complex variables deals exclusively with such functions.

About half a century ago, electrical engineers, and then many more applied mathematicians, have discovered that some bounded analytic functions as our F_p , or its modification in (15), can be interpreted as transfer functions of multivariate, linear systems appearing in control theory. The second part of our article contains an introduction aimed at the non-expert to the realization theory of bounded analytic functions in the polydisk. We give precise references to recent and classical works and we illustrate the benefits of this new dictionary with examples arising in the construction of tight wavelet frames.

In particular we show in Theorem 4.2 that polynomials $p \in \mathbb{C}[z]$ with non-negative coefficients and satisfying the conditions (3) and (6) are transfer functions of finite dimensional linear systems, complementing and improving typical wavelet theory results [22]. Moreover, we use the adjunction formula for transfer functions, Proposition 5.7, in order to devise a new technique for passing from the sos-decomposition in (7) to the construction of a_1, \dots, a_N in the UEP (4). In Example 4.7 we show that, even for the simplest nonseparable mask of the piecewise linear three-directional box-spline B_{111} , the techniques from system theory improve all known frame constructions. Indeed, we obtain 5 trigonometric polynomials a_1, \dots, a_5 of coordinate degree 2, which complement the mask of B_{111} .

Acknowledgement. The second author is indebted to the Gambrinus Fellowship of Technische Universität Dortmund for support and hospitality in June 2013. The present work could not be finished without the generous support of the Institute of Mathematics and Applications in Singapore, where all authors met in December 2013.

2 Convexity properties of tight wavelet frames

In this section we study the properties of the sets of trigonometric polynomials satisfying the sub-QMF condition and its subset of trigonometric polynomials which yield tight wavelet frames.

Denote by \mathbb{T}^d the d -dimensional torus

$$\mathbb{T}^d = \{z = (z_1, \dots, z_d) \in \mathbb{C}^d : |z_j| = 1 \text{ for } j = 1, \dots, d\}.$$

The vector space $\mathbb{C}[\mathbb{T}^d]$ of trigonometric polynomials on \mathbb{T}^d is equipped with the finest locally convex topology under which all linear functionals are continuous. A basis of neighborhoods of the origin is defined by the seminorms

$$|p|_\lambda = |\lambda(p)|, \quad \lambda \in (\mathbb{C}[\mathbb{T}^d])^*, \quad p \in \mathbb{C}[\mathbb{T}^d].$$

For the dilation matrix $M \in \mathbb{Z}^{d \times d}$ define $m := |\det(M)| \neq 0$. The translation group $G = 2\pi M^{-T} \mathbb{Z}^d / 2\pi \mathbb{Z}^d$ acts on $p \in \mathbb{C}[\mathbb{T}^d]$ by

$$p^\sigma(z) := p(e^{-i\sigma_1} z_1, \dots, e^{-i\sigma_d} z_d), \quad z \in \mathbb{T}^d, \quad \sigma \in G.$$

Let $G' := G^*$ be the character group of G and $p = \sum_{\chi \in G'} p_\chi$ be the isotypical decomposition of p . For each $\chi \in G'$, we choose $\alpha_\chi \in \mathbb{Z}^d$ such that $\tilde{p}_\chi = z^{\alpha_\chi} p_\chi$ is G -invariant. In signal analysis, \tilde{p}_χ is called polyphase component

of p . Then we have

$$\sum_{\sigma \in G} p^{\sigma*} p^{\sigma} = m \sum_{\chi \in G'} p_{\chi}^* p_{\chi} = m \sum_{\chi \in G'} \tilde{p}_{\chi}^* \tilde{p}_{\chi} \quad \text{on } \mathbb{T}^d. \quad (8)$$

It is well-known that the set of all real-valued, non-negative trigonometric polynomials is a closed convex cone which is also closed under multiplication. We show that the set

$$\mathcal{S} = \mathcal{S}(M, d) = \{p \in \mathbb{C}[\mathbb{T}^d] : p(1, \dots, 1) = 1, f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^{\sigma} \geq 0 \text{ on } \mathbb{T}^d\}$$

of trigonometric polynomials that satisfy the sub-QMF condition has the same properties. Its extremal points are those $p \in \mathbb{C}[\mathbb{T}^d]$ with $f = 0$ on \mathbb{T}^d . Clearly, the set \mathcal{S} is not compact.

Our main interest lies in the set

$$\mathcal{S}_{\text{sos}} = \{p \in \mathcal{S} : f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^{\sigma} \text{ is hermitean sum of squares on } \mathbb{T}^d\},$$

which is the subset of \mathcal{S} of those p that allow the sos-representation (7) of f . By [5, Theorem 2.2] or [22, Theorem 3.4], the trigonometric polynomials $p \in \mathcal{S}_{\text{sos}}$ yield tight wavelet frames in (5). We show that \mathcal{S}_{sos} is convex, is not closed for $d \geq 3$, and has the same extremal points as \mathcal{S} . Moreover, the set \mathcal{S}_{sos} is closed under multiplication. This latter property allows us to provide upper bounds on the number of the frame generators for box-splines of any dimension, and to improve known upper bounds for special types of bivariate and trivariate box-splines in [21, 22].

By the Riesz-Fejer lemma ($d = 1$) and Scheiderer's result [24] ($d = 2$), we have $\mathcal{S}_{\text{sos}} = \mathcal{S}$ for $d = 1, 2$, while the example in [5] shows $\mathcal{S}_{\text{sos}} \subsetneq \mathcal{S}$ for $d \geq 3$. The following result describes the properties of the set \mathcal{S}_{sos} also for dimensions $d \geq 3$.

Theorem 2.1. *Let $d \in \mathbb{N}$.*

- (i) *The set \mathcal{S} is closed and convex. Moreover, \mathcal{S} is closed under multiplication.*
- (ii) *The set \mathcal{S}_{sos} is convex and is closed under multiplication.*

Proof. Closedness of the set \mathcal{S} is obvious. To show the convexity of the sets \mathcal{S} and \mathcal{S}_{sos} , let first $p_1, p_2 \in \mathcal{S}$, $t \in (0, 1)$ and set $q := (1 - t)p_1 + tp_2$,

$u := p_1 - p_2$. The identity

$$\begin{aligned} q^*q + t(1-t)u^*u &= (1-t)^2 p_1^* p_1 + t^2 p_2^* p_2 + t(1-t)(p_1^* p_2 + p_2^* p_1) \\ &\quad + t(1-t)(p_1^* p_1 + p_2^* p_2 - p_1^* p_2 - p_2^* p_1) \\ &= (1-t)p_1^* p_1 + t p_2^* p_2 \end{aligned}$$

implies

$$\sum_{\sigma \in G} q^{*\sigma} q^\sigma + t(1-t) \sum_{\sigma \in G} u^{*\sigma} u^\sigma = (1-t) \sum_{\sigma \in G} p_1^{*\sigma} p_1^\sigma + t \sum_{\sigma \in G} p_2^{*\sigma} p_2^\sigma,$$

and, thus, we have

$$1 - \sum_{\sigma \in G} q^{*\sigma} q^\sigma = (1-t) \left(1 - \sum_{\sigma \in G} p_1^{*\sigma} p_1^\sigma \right) + t \left(1 - \sum_{\sigma \in G} p_2^{*\sigma} p_2^\sigma \right) + t(1-t) \sum_{\sigma \in G} u^{*\sigma} u^\sigma. \quad (9)$$

Therefore, $q \in \mathcal{S}$. Furthermore, if $p_1, p_2 \in \mathcal{S}_{\text{sos}}$, then $q \in \mathcal{S}_{\text{sos}}$.

Furthermore, we have

$$1 - \sum_{\sigma \in G} (p_1 p_2)^{\sigma*} (p_1 p_2)^\sigma = 1 - \sum_{\sigma \in G} p_1^{\sigma*} p_1^\sigma + \sum_{\sigma \in G} p_1^{\sigma*} p_1^\sigma (1 - p_2^{\sigma*} p_2^\sigma) \geq 0. \quad (10)$$

This implies that both \mathcal{S} and \mathcal{S}_{sos} are closed under multiplication. \square

We next characterize the extremal points of \mathcal{S} and \mathcal{S}_{sos} .

Theorem 2.2. *Let $d \in \mathbb{N}$.*

(i) *The trigonometric polynomial $p \in \mathcal{S}_{\text{sos}}$ is an extremal point of \mathcal{S}_{sos} if and only if p satisfies the QMF-condition*

$$\sum_{\sigma \in G} p^{\sigma*} p^\sigma \equiv 1 \quad \text{on } \mathbb{T}^d. \quad (11)$$

(ii) *The extremal points of \mathcal{S} and \mathcal{S}_{sos} coincide.*

Proof. Proof of (i): On the one hand, let $q \in \mathcal{S}_{\text{sos}}$ satisfy the QMF-condition and assume $q = (1-t)p_1 + tp_2$ with $p_1, p_2 \in \mathcal{S}_{\text{sos}}$, $t \in (0, 1)$. From (9) we conclude that both p_1 and p_2 satisfy the QMF-condition in (11), and $u = p_1 - p_2 = 0$. Therefore, $q = p_1$ is an extremal point.

On the other hand, let $p \in \mathcal{S}_{\text{sos}}$ be an extremal point and let $r = 1 - \sum_{\sigma \in G} p^{\sigma*} p^\sigma$. The isotypical components p_χ of p , $\chi \in G'$, satisfy

$$m \sum_{\chi \in G'} p_\chi^* p_\chi = \sum_{\sigma \in G} p^{\sigma*} p^\sigma \leq 1.$$

We define the trigonometric polynomials p_+ and p_- by their isotypical components

$$p_{\pm,0} = p_0 \pm \frac{r}{4m}, \quad p_{\pm,\chi} = p_\chi, \quad \chi \neq 0. \quad (12)$$

Note that, indeed, $p_0 \pm \frac{r}{4m}$ defines an isotypical component, since both p_0 and r are G -invariant. We next show that p_\pm belong to \mathcal{S}_{sos} . Note that

$$\begin{aligned} 1 - m \sum_{\chi \in G'} p_{+, \chi}^* p_{+, \chi} &= 1 - m \sum_{\chi \in G'} p_\chi^* p_\chi - \frac{r}{2} \text{Re}(p_0) - \frac{r^2}{16m} \\ &= r \left(1 - \frac{1}{2} \text{Re}(p_0) - \frac{r}{16m} \right). \end{aligned}$$

By definition of r , we have $0 \leq r \leq 1$. Thus, $(16m)^{-1}r < 2^{-1}$ and $p_0^* p_0 \leq 1$, which imply

$$1 - \frac{1}{2} \text{Re}(p_0) - \frac{r}{16m} \geq \frac{1}{2} - \frac{r}{16m} > 0 \quad \text{on } \mathbb{T}^d.$$

This strict positivity, by [25], and the assumption that r is sos yield that $p_+ \in \mathcal{S}_{\text{sos}}$. Analogously, $p_- \in \mathcal{S}_{\text{sos}}$. Since $p = \frac{1}{2}(p_+ + p_-)$ is extremal, we conclude that $p = p_+ = p_-$ and therefore $r = 0$. This shows that p satisfies the QMF condition.

The proof that \mathcal{S} has the same extremal points is similar to the proof of part (i). \square

Remark 2.3. We have seen that the two sets $\mathcal{S}_{\text{sos}} \subseteq \mathcal{S}$ are both convex and their respective extremal points are the same, yet the inclusion is proper for $d > 2$ [5]. The following remarks are an attempt to better understand the geometry of \mathcal{S}_{sos} and \mathcal{S} .

According to the Krein-Milman theorem, any compact convex subset of $\mathbb{C}[\mathbb{T}^d]$ is the closed convex hull of its extremal points. Although the convex set \mathcal{S} is closed, it is easy to see that \mathcal{S} is not compact. So there is no reason to expect that \mathcal{S} agrees with the closed convex hull of its extremal points.

For $d > 2$, the convex subset \mathcal{S}_{sos} of \mathcal{S} fails to be closed. Indeed, let \mathcal{S}' be the subset of \mathcal{S} consisting of all $p \in \mathbb{C}[\mathbb{T}^d]$ for which $p(1, \dots, 1) = 1$ and

$$f(z) = 1 - \sum_{\sigma \in G} p^{\sigma*}(z) p^\sigma(z) > 0 \quad , z \in \mathbb{T}^d \setminus G,$$

and for which the Hessian of f at $1, \dots, 1$ is positive definite. Then $\mathcal{S}' \subseteq \mathcal{S}_{\text{sos}}$ by [5, Theorem 3.2]. But the closure of \mathcal{S}' is not contained in \mathcal{S}_{sos} . To see this, one can modify the construction of [5, Theorem 2.5]: Let

$$p_t(z) = \left(1 - t(y_1^2 + y_2^2 + y_3^2) - c \cdot m(z) \right) a(z)$$

where c and t are small positive real numbers and $y_j, m(z), a(z)$ are as in [5]. When t is small and positive, p_t lies in \mathcal{S}' . But for $t = 0$ we have $p_t \notin \mathcal{S}_{\text{sos}}$.

3 Bounds on the number of frame generators for box splines

We use the closedness under multiplication of \mathcal{S}_{sos} to improve the upper bound in [21, 22] for the number N of the frame generators for box splines. The explicit upper bound for N , see [5, 22], depends on the length of the sos decomposition of f and on m .

Let $p \in \mathbb{C}[\mathbb{T}^d]$. Denote by $L(p)$ the hermitian sos length of $f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^\sigma$, i.e., the smallest number r such that $f = |h_1|^2 + \dots + |h_r|^2$ with G -invariant trigonometric polynomials $h_j \in \mathbb{C}[\mathbb{T}^d]$. Also, let $\ell(p)$ be the sos length of $1 - p^* p$. As usual we set these numbers equal to ∞ , if the respective polynomials are not sos. Note that $L(p) < \infty$ implies $\ell(p) \leq L(p) + m - 1 < \infty$.

We first prove two auxiliary lemmas.

Lemma 3.1. *Let $p, q \in \mathcal{S}$. Then*

$$L(pq) \leq L(p) + m\ell(q).$$

Proof. Note that we only need to prove the claim in the case $L(p) < \infty$ and $\ell(q) < \infty$. Then, trivially,

$$1 - q^* q = \sum_{j=1}^{\ell(q)} \tau_j^* \tau_j \quad \text{on } \mathbb{T}^d$$

and also, for $\sigma \in G$,

$$1 - q^{\sigma*} q^\sigma = \sum_{j=1}^{\ell(q)} \tau_j^{\sigma*} \tau_j^\sigma \quad \text{on } \mathbb{T}^d.$$

Note that

$$\sum_{\sigma \in G} p^{\sigma*} p^\sigma (1 - q^{\sigma*} q^\sigma) = \sum_{j=1}^{\ell(q)} \sum_{\sigma \in G} (p\tau_j)^{\sigma*} (p\tau_j)^\sigma = m \sum_{j=1}^{\ell(q)} \sum_{\chi \in G'} (\widetilde{p\tau_j})_\chi^* (\widetilde{p\tau_j})_\chi$$

has a G -invariant sos of length $m\ell(q)$. Thus, the claim follows from

$$1 - \sum_{\sigma \in G} (pq)^{\sigma*} (pq)^\sigma = \left(1 - \sum_{\sigma \in G} p^{\sigma*} p^\sigma\right) + \sum_{\sigma \in G} p^{\sigma*} p^\sigma (1 - q^{\sigma*} q^\sigma). \quad (13)$$

□

Lemma 3.2. *Assume that G_1, \dots, G_r are subgroups of G such that the product map $G_1 \times \dots \times G_r \rightarrow G$ is bijective. Assume further that for every $j = 1, \dots, r$, a polynomial $p_j \in \mathbb{C}[\mathbb{T}^d]$ is G_k -invariant for all $k \neq j$ and is such that $1 - \sum_{\sigma \in G_j} |p_j^\sigma|^2$ is sos. Then $p := p_1 \dots p_r \in \mathcal{S}_{\text{sos}}$.*

Proof. By assumption we have $p^\sigma = p_1^{\sigma_1} \dots p_r^{\sigma_r}$ whenever $\sigma_j \in G_j$, $j = 1, \dots, r$, and $\sigma = \sigma_1 \dots \sigma_r$. Therefore,

$$1 - \sum_{\sigma \in G} |p^\sigma|^2 = 1 - \sum_{\sigma_1 \in G_1} \dots \sum_{\sigma_r \in G_r} |p_1^{\sigma_1} \dots p_r^{\sigma_r}|^2 = 1 - \prod_{j=1}^r \left(\sum_{\sigma_j \in G_j} |p_j^{\sigma_j}|^2 \right).$$

Denoting $t_j := \sum_{\sigma_j \in G_j} |p_j^{\sigma_j}|^2$, $j = 1, \dots, r$, we get that

$$1 - \sum_{\sigma \in G} |p^\sigma|^2 = 1 - t_1 \dots t_r = \sum_{j=1}^r t_1 \dots t_{j-1} (1 - t_j) \quad (14)$$

is sos. □

We are finally ready to derive the upper bound for the number of the frame generators for box splines. From now on we assume that $M = 2I$, hence $G \cong \pi\{0, 1\}^d$.

Proposition 3.3. *Let*

$$p = \prod_{j=1}^r \left(\frac{1 + z^{\theta_j}}{2} \right)^{\ell_j}, \quad \theta_j \in \mathbb{Z}^d, \quad \ell_j \in \mathbb{N},$$

and assume that $\theta_1, \dots, \theta_d$ span \mathbb{Z}^d modulo $2\mathbb{Z}^d$. Then

$$L(p) \leq d + (r - d)2^d.$$

Proof. Define polynomials

$$p_j(z) = \left(\frac{1 + z^{\theta_j}}{2} \right)^{\ell_j}, \quad j = 1, \dots, r,$$

which we each treat as a univariate polynomial in the variable $u_j := z^{\theta_j}$, respectively. We first show that $L(p_1 \cdots p_d) = d$. For $\theta_j \in \mathbb{Z}^d$ define $\bar{\theta}_j = \theta_j + 2\mathbb{Z}^d \in \mathbb{Z}^d/2\mathbb{Z}^d$. By assumption, $\bar{\theta}_1, \dots, \bar{\theta}_d$ is a basis of $\mathbb{Z}^d/2\mathbb{Z}^d$. Let $b_1, \dots, b_d \in \mathbb{Z}^d$ such that $\bar{b}_1, \dots, \bar{b}_d$ is the dual basis of $\mathbb{Z}^d/2\mathbb{Z}^d$. For $j = 1, \dots, d$ let $G_j \subseteq G$ be the subgroup of order two generated by πb_j . Then the group G is the direct product of G_1, \dots, G_d . Moreover, for $j \neq k$ and $\sigma \in G_k$, we have $e^{i\sigma \cdot \theta_j} = 1$ and, thus, the polynomial p_j is invariant under G_k . Note next that the non-negative polynomials

$$t_j := \sum_{\sigma \in G_j} |p_j^\sigma|^2 = \left| \frac{1 + z^{\theta_j}}{2} \right|^{2\ell_j} + \left| \frac{1 - z^{\theta_j}}{2} \right|^{2\ell_j}, \quad j = 1, \dots, d,$$

and $1 - t_j$ are G -invariant. By the Féjer-Riesz Lemma, t_j and $1 - t_j$, $j = 1, \dots, d$, are therefore single G -invariant squares in the variable u_j . Therefore, Lemma 3.2 implies that $q = p_1 \cdots p_d \in \mathcal{S}_{sos}$ and, by (14), $L(q) = d$.

Next, note that the Riesz-Fejer Lemma implies that $\ell(p_j) = 1$, $j = d + 1, \dots, r$. Thus, by Lemma 3.1, in particular by identity (13), we get $L(qp_{d+1}) \leq L(q) + 2^d$, where $|G| = 2^d$. The claim follows then by induction on n for the polynomials $q \prod_{j=d+1}^n p_j$, $n = d + 2, \dots, r$. \square

Remark 3.4. By the constructive algorithm in [22] and by Proposition 3.3, for $d = 2$ and $r = 2, 3, 4, 5, \dots$, we get the upper bounds 6, 10, 14, 18... for the number of tight frame generators for the corresponding r -directional box splines. This improves the previously known upper bounds from [21], namely 11, 19, for $d = 2$ and $r = 3, 4$. Note that our upper bounds are not sharp, in general. For example, for

$$p(z_1, z_2) = \left(\frac{1 + z_1}{2} \right) \left(\frac{1 + z_2}{2} \right) \left(\frac{1 + z_1 z_2}{2} \right),$$

i.e. $d = 2$ and $r = 3$, a tight wavelet frame with only 6 frame generators was constructed in [22], and in subsection 4.3 we construct a tight wavelet frame with only 5 frame generators.

4 System theory and wavelet tight frames

In this section we establish a connection between constructions of tight wavelet frames and some fundamental results from system theory. For the

reader's convenience, we include an overview of the relevant results from system theory in section 5.

Here, instead of working with trigonometric polynomials, we consider algebraic polynomials $p \in \mathbb{C}[z]$. We write $M = (m_1, \dots, m_d) \in \mathbb{Z}^{d \times d}$ and define the isotypical components p_χ and the polyphase components \tilde{p}_χ , $\chi \in G'$, similarly to (8). Hence, the polyphase components of $p = \sum_{\beta \in \mathbb{Z}^d} p(\beta) z^\beta$ are

$$\tilde{p}_\chi = z^{-\alpha_\chi} p_\chi = \sum_{\beta \in \mathbb{Z}^d} p(\alpha_\chi + M\beta) z^{M\beta}.$$

Therefore, we consider \tilde{p}_χ as polynomials in the variable $\xi = z^M := (z^{m_1}, \dots, z^{m_d})$, due to the identity $\xi^\beta = z^{M\beta}$.

Define the vector-valued analytic function $f_p : \mathbb{C}^d \rightarrow \mathbb{C}^m$ by

$$f_p(\xi) = (m^{1/2} \tilde{p}_\chi(\xi))_{\chi \in G'}. \quad (15)$$

Then, in the polarized version with variables $\xi, \eta \in \mathbb{C}^d$, we have

$$1 - m \sum_{\chi \in G'} \tilde{p}_\chi(\eta)^* \tilde{p}_\chi(\xi) = 1 - f_p(\eta)^* f_p(\xi).$$

We assume that the analytic function f_p satisfies $\|f_p(\xi)\| \leq 1$ for all ξ in the polydisk

$$\mathbb{D}^d = \left\{ \xi = (\xi_1, \dots, \xi_d) \in \mathbb{C}^d : |\xi_j| < 1, j = 1, \dots, d \right\}.$$

If $\|f_p(\xi)\| = 1$ on \mathbb{T}^d , then f_p is called *inner*. The requirement that either $\|f_p(\xi)\| \leq 1$ or $\|f_p(\xi)\| = 1$ states that the trigonometric polynomial $p|_{\mathbb{T}^d}$ satisfies either the sub-QMF (6) or QMF (11) condition, respectively.

It is then natural to ask, if such functions f_p possess the decomposition

$$1 - f_p(\eta)^* f_p(\xi) = q_0(\eta)^* q_0(\xi) + \sum_{j=1}^d (1 - \xi_j \bar{\eta}_j) q_j(\eta)^* q_j(\xi) \quad (16)$$

with polynomial maps $q_j : \mathbb{C}^d \rightarrow \mathbb{C}^{N_j}$, $N_j \in \mathbb{N}$ and $j = 0, \dots, d$. If we consider $\xi = \eta \in \mathbb{T}^d$ in (16), then the last sum disappears, and with $q_0 = (h_1, \dots, h_{N_0})^T$, we obtain the sos decomposition in (7) with G -invariant trigonometric polynomials h_1, \dots, h_{N_0} ,

$$1 - \sum_{\sigma \in G} p^{\sigma*} p^\sigma = 1 - f_p(\xi)^* f_p(\xi) = \sum_{j=1}^{N_0} h_j(\xi)^* h_j(\xi), \quad \xi = z^M.$$

In other words, by construction of the decomposition (16) on \mathbb{D}^d , we prove that the trigonometric polynomial $p|_{\mathbb{T}^d}$ is in \mathcal{S}_{sos} and, in addition, we have the sos-decomposition (7) of sos-length N_0 . Moreover, we connect the bilinear decomposition (16) with the realization formula

$$\begin{pmatrix} f_p \\ q_0 \end{pmatrix}(\xi) = A + BE(\xi)(I - DE(\xi))^{-1}C, \quad \xi \in \mathbb{D}^d,$$

in Theorem 5.3(c) and obtain a parameterized version (in terms of the isometry $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$) of the decomposition (7).

This motivates us to study the properties of the set

$$\mathcal{S}_A := \{p \in \mathbb{C}[z] : f_p \text{ satisfies (16)}\}.$$

Note that, for $p \in \mathcal{S}_A$, the function $(f_p, q_0)^T : \mathbb{C}^d \rightarrow \mathbb{C}^{m+N_0}$ is inner and in the Schur-Agler class, see Theorem 5.3 with $X = Y = \mathbb{C}$.

Unfortunately, if the corresponding trigonometric polynomial $p|_{\mathbb{T}^d}$ is in \mathcal{S}_{sos} , then we do not necessarily have $p \in \mathcal{S}_A$. The following example illustrates this observation.

Example 4.1. Let $g(z) = z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3$, $z \in \mathbb{D}^3$, be the polynomial in (35). The maxima of $|g|$ are at

$$z_1 = z_2 = e^{2\pi i/3} z_3, \quad z_3 \in \mathbb{T},$$

and

$$z_1 = z_2 = e^{-2\pi i/3} z_3, \quad z_3 \in \mathbb{T},$$

and permutations thereof. We select

$$z_1 = z_2 = e^{2\pi i/3}, \quad z_3 = 1,$$

where $|g(z_1, z_2, z_3)| = \|g\|_{\infty, \mathbb{D}^3} = 3\sqrt{3}$. We define another polynomial

$$q(z) = \frac{g(e^{2\pi i/3} z_1, e^{2\pi i/3} z_2, z_3)}{g(e^{2\pi i/3}, e^{2\pi i/3}, 1)} = \frac{1}{3(1 + e^{\pi i/3})} (z_1^3 + z_2^3 + z_3^3 + 3e^{\pi i/3} z_1 z_2 z_3)$$

so that $q(1, \dots, 1) = 1$, $\|q\|_{\infty, \mathbb{D}^3} = 1$ and $\|q(T_1, T_2, T_3)\| = \frac{2}{\sqrt{3}}$ with the appropriately rotated commutative contractions T_1 , T_2 and T_3 for which $\|g(T_1, T_2, T_3)\| = 6$, see subsection 5.2. Next, for the dilation matrix $M = 2I$, we define

$$\xi = (\xi_1, \xi_2, \xi_3) = (z_1^2, z_2^2, z_3^2) \in \mathbb{D}^3$$

and the polynomial $p \in \mathbb{C}[z]$ by

$$p(z) = 8^{-1}q(z^2) \sum_{\chi \in G'} z^{\alpha_\chi}, \quad z \in \mathbb{D}^3, \quad \alpha_\chi \in \Gamma = \{0, 1\}^3.$$

The corresponding column vector $f_p : \mathbb{C}^3 \rightarrow \mathbb{C}^8$ of the polyphase components of p is given by

$$f_p(\xi) = 8^{-1/2}q(\xi) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

The polynomial p satisfies $p(1, \dots, 1) = 1$, and

$$1 - f_p(\eta)^* f_p(\xi) = 1 - q(\eta)^* q(\xi)$$

does not possess the representation in (16), i.e. $p \notin \mathcal{S}_A$. We show next that $p|_{\mathbb{T}^3} \in \mathcal{S}_{sos}$. Note that if we dehomogenize

$$q(z^2) = \frac{z_3^6}{3(1 + e^{\pi i/3})} \left(\frac{z_1^6}{z_3^6} + \frac{z_2^6}{z_3^6} + 1 + 3e^{\pi i/3} \frac{z_1^2 z_2^2}{z_3^4} \right), \quad z \in \mathbb{T}^3,$$

and set

$$y_1 = \frac{z_1^2}{z_3^2} \quad \text{and} \quad y_2 = \frac{z_2^2}{z_3^2},$$

then the polynomial $1 - q^*q$ in the variables y_1 and y_2 is a 2-dim non-negative polynomial on \mathbb{T}^2 . Thus, by [24], $p|_{\mathbb{T}^3} \in \mathcal{S}_{sos}$.

4.1 Polynomials p with non-negative coefficients

Despite the difficulties illustrated in example 4.1, we are able to describe a large class of analytic polynomials p which belong to the set \mathcal{S}_A .

Theorem 4.2. *Let $p \in \mathbb{C}[z]$ have non-negative coefficients, and let its polyphase components satisfy $\tilde{p}_\chi(1, \dots, 1) = m^{-1}$, $\chi \in G'$. Then $p \in \mathcal{S}_A$.*

Proof. Let $\Gamma \subseteq \mathbb{N}_0^d$ be a set of representatives of G' . Define the sets

$$\mathcal{I} = \{\alpha \in \mathbb{Z}^d : p(\alpha) \neq 0\} \quad \text{and} \quad \tilde{\mathcal{I}} = \{\alpha \in \mathbb{Z}^d : \exists \gamma \in \Gamma \text{ such that } \gamma + M\alpha \in \mathcal{I}\}.$$

Assume that the index set $\tilde{\mathcal{I}}$ is linearly ordered, e.g. by the lexicographical ordering. Also, for $j = 1, \dots, d$, we define

$$n_j = \max\{\alpha_j : \alpha = (\alpha_1, \dots, \alpha_d) \in \tilde{\mathcal{I}}\}$$

and

$$\mathcal{I}_j = \{0, \dots, n_1\} \times \dots \times \{0, \dots, n_j\}. \quad (17)$$

Furthermore, we define the row and column vectors

$$\mathbf{p}_\chi = (p(\alpha_\chi + M\alpha) : \alpha \in \tilde{\mathcal{I}}), \quad \chi \in G', \quad \text{and} \quad v(\xi) = (\xi^\alpha : \alpha \in \tilde{\mathcal{I}})^T,$$

respectively. Define also the monomial vector

$$v_j(\xi_1, \dots, \xi_d) = (\xi_1^{\beta_1} \dots \xi_j^{\beta_j} : \beta = (\beta_1, \dots, \beta_j) \in \mathcal{I}_j)^T. \quad (18)$$

Note that $v(\eta)^* = (\bar{\eta}^\alpha : \alpha \in \tilde{\mathcal{I}})$ and $v_j(\eta)^* = (\bar{\eta}^\alpha : \alpha \in \mathcal{I}_j)$ are then row vectors.

Next we write the polyphase components of p in the vector form

$$\tilde{p}_\chi(\xi) = \sum_{\alpha \in \tilde{\mathcal{I}}} p(\alpha_\chi + M\alpha) \xi^\alpha = \mathbf{p}_\chi \cdot v(\xi), \quad \xi \in \mathbb{C}^d.$$

By Theorem 5.3, it suffices to show that

$$1 - m \sum_{\chi \in G'} v(\eta)^* \mathbf{p}_\chi^* \mathbf{p}_\chi v(\xi) = v(\eta)^* A_0 v(\xi) + \sum_{j=1}^d (1 - \xi_j \bar{\eta}_j) v_j(\eta)^* A_j v_j(\xi) \quad (19)$$

for $\xi, \eta \in \mathbb{D}^d$ with hermitean positive semi-definite matrices A_j , $j = 0, \dots, d$. Due to $\tilde{p}_\chi(1, \dots, 1) = m^{-1}$, we have

$$1 = \sum_{\chi \in G'} \sum_{\alpha \in \tilde{\mathcal{I}}} p(\alpha_\chi + M\alpha).$$

Thus, we get

$$\begin{aligned} 1 - m \sum_{\chi \in G'} v(\eta)^* \mathbf{p}_\chi^* \mathbf{p}_\chi v(\xi) &= 1 - m \sum_{\chi \in G'} \sum_{\alpha, \beta \in \tilde{\mathcal{I}}} p(\alpha_\chi + M\alpha) p(\alpha_\chi + M\beta) \xi^\alpha \bar{\eta}^\beta \\ &= \sum_{\chi \in G'} \left(\sum_{\alpha \in \tilde{\mathcal{I}}} p(\alpha_\chi + M\alpha) - m \sum_{\alpha \in \tilde{\mathcal{I}}} p(\alpha_\chi + M\alpha)^2 \xi^\alpha \bar{\eta}^\alpha \right. \\ &\quad \left. - m \sum_{\substack{\alpha, \beta \in \tilde{\mathcal{I}} \\ \alpha \neq \beta}} p(\alpha_\chi + M\alpha) p(\alpha_\chi + M\beta) \xi^\alpha \bar{\eta}^\beta \right) \\ &= \sum_{\chi \in G'} \left(\sum_{\alpha \in \tilde{\mathcal{I}}} (p(\alpha_\chi + M\alpha) - m p(\alpha_\chi + M\alpha)^2) \xi^\alpha \bar{\eta}^\alpha \right. \\ &\quad \left. - m \sum_{\substack{\alpha, \beta \in \tilde{\mathcal{I}} \\ \alpha \neq \beta}} p(\alpha_\chi + M\alpha) p(\alpha_\chi + M\beta) \xi^\alpha \bar{\eta}^\beta + \sum_{\alpha \in \tilde{\mathcal{I}}} (1 - \xi^\alpha \bar{\eta}^\alpha) p(\alpha_\chi + M\alpha) \right). \end{aligned}$$

Define the $|\tilde{\mathcal{I}}| \times |\tilde{\mathcal{I}}|$ matrices $A_{\chi,0}$, $\chi \in G'$, by

$$A_{\chi,0}(\alpha, \beta) = \begin{cases} p(\alpha_\chi + M\alpha) - mp(\alpha_\chi + M\alpha)^2, & \text{if } \alpha = \beta, \\ -mp(\alpha_\chi + M\alpha)p(\alpha_\chi + M\beta), & \text{otherwise,} \end{cases} \quad \alpha, \beta \in \tilde{\mathcal{I}}.$$

The simple observation

$$p(\alpha_\chi + M\alpha) = p(\alpha_\chi + M\alpha) m \tilde{p}_\chi(1, \dots, 1) = m p(\alpha_\chi + M\alpha) \sum_{\beta \in \tilde{\mathcal{I}}} p(\alpha_\chi + M\beta)$$

implies that $A_{\chi,0}$, $\chi \in G'$, are weakly diagonally dominant and, thus, are positive semi-definite. Therefore,

$$1 - m \sum_{\chi \in G'} v(\eta)^* \mathbf{p}_\chi^* \mathbf{p}_\chi v(\xi) = v(\eta)^* A_0 v(\xi) + \sum_{\alpha \in \tilde{\mathcal{I}}} (1 - \xi^\alpha \bar{\eta}^\alpha) \sum_{\chi \in G'} p(\alpha_\chi + M\alpha)$$

with the positive semi-definite matrix $A_0 = \sum_{\chi \in G'} A_{\chi,0}$. For $j = 1, \dots, d$ and $\beta \in \mathcal{I}_j$, we also define

$$\mathcal{I}(\beta) = \{\alpha \in \tilde{\mathcal{I}} : \alpha_k = \beta_k \text{ for } 1 \leq k \leq j-1, \alpha_j > \beta_j\}. \quad (20)$$

Then, due to

$$1 - \eta^\alpha = 1 - \eta_1^{\alpha_1} + \eta_1^{\alpha_1} (1 - \eta_2^{\alpha_2}) + \dots + \eta_1^{\alpha_1} \dots \eta_{d-1}^{\alpha_{d-1}} (1 - \eta_d^{\alpha_d}), \quad \eta \in \mathbb{C}^d, \alpha \in \mathbb{N}_0^d,$$

and

$$1 - \eta_j^{\alpha_j} = (1 - \eta_j) \sum_{k=0}^{\alpha_j-1} \eta_j^k, \quad \alpha_j > 0,$$

we obtain

$$\begin{aligned} \sum_{\alpha \in \tilde{\mathcal{I}}} (1 - \xi^\alpha \bar{\eta}^\alpha) \sum_{\chi \in G'} p(\alpha_\chi + M\alpha) = \\ \sum_{j=1}^d (1 - \xi_j \bar{\eta}_j) \sum_{\beta \in \mathcal{I}_j} (\xi_1, \dots, \xi_j)^\beta (\bar{\eta}_1, \dots, \bar{\eta}_j)^\beta \sum_{\alpha \in \mathcal{I}(\beta)} \sum_{\chi \in G'} p(\alpha_\chi + M\alpha). \end{aligned}$$

Then, we get

$$\sum_{\alpha \in \tilde{\mathcal{I}}} (1 - \xi^\alpha \bar{\eta}^\alpha) \sum_{\chi \in G'} p(\alpha_\chi + M\alpha) = \sum_{j=1}^d (1 - \xi_j \bar{\eta}_j) v_j(\eta)^* A_j v_j(\xi), \quad (21)$$

with diagonal $|\mathcal{I}_j| \times |\mathcal{I}_j|$ matrices A_j whose non-negative diagonal entries are $A_j(\beta, \beta) = \sum_{\alpha \in \mathcal{I}(\beta)} \sum_{\chi \in G'} p(\alpha_\chi + M\alpha)$, $\beta \in \mathcal{I}_j$. \square

Remark 4.3. Note that the matrices A_0 and A_j , $j = 1, \dots, d$, in (21) define the polynomial maps q_j , $j = 0, \dots, d$, in (16) by

$$q_0(\xi) = \sqrt{A_0}v(\xi) \quad \text{and} \quad q_j(\xi) = \sqrt{A_j}v_j(\xi). \quad (22)$$

Since A_j is diagonal and the entries of v_j belong to $\mathbb{C}[\xi_1, \dots, \xi_j]$, the vector $q_j(\xi)$ is a vector of (scaled) monomials in $\mathbb{C}[\xi_1, \dots, \xi_j]$, $j = 1, \dots, d$.

Theorem 4.2 and Theorem 5.3 imply that the maps f_p and q_0 satisfy

$$\begin{pmatrix} f_p \\ q_0 \end{pmatrix}(\xi) = A + BE(\xi)(I - DE(\xi))^{-1}C, \quad \xi \in \mathbb{D}^d, \quad (23)$$

with an isometry

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{array}{c} \mathbb{C} \\ \oplus \\ \mathbb{C}^{|\mathcal{I}_1|+\dots+|\mathcal{I}_d|} \end{array} \longrightarrow \begin{array}{c} \mathbb{C}^{m+r} \\ \oplus \\ \mathbb{C}^{|\mathcal{I}_1|+\dots+|\mathcal{I}_d|} \end{array}, \quad r = |\tilde{\mathcal{I}}|,$$

and the block diagonal matrix

$$E(\xi) = \text{diag}(I_{|\mathcal{I}_1|}\xi_1, \dots, I_{|\mathcal{I}_d|}\xi_d). \quad (24)$$

To obtain the contractive representation for f_p one just deletes the last r rows of A and B and leaves C and D unchanged. In the proof of the following Corollary we define one possible choice of the matrices A , B , C and D and study the properties of D .

Corollary 4.4. *Under assumptions of Theorem 4.2, there exists a realization*

$$\begin{pmatrix} f_p \\ q_0 \end{pmatrix}(\xi) = A + BE(\xi)(I - DE(\xi))^{-1}C, \quad \xi \in \mathbb{D}^d,$$

with nilpotent matrix $DE(\xi)$.

Proof. Let the matrices A_j , $0 \leq j \leq d$, be defined as in the proof of Theorem 4.2, the sets \mathcal{I}_j and column vectors v_j be as in (17) and (18) (lexicographical ordering), respectively. Then, by Theorem 4.2, we have

$$1 - f_p(\eta)^* f_p(\xi) - q_0(\eta)^* q_0(\xi) = \sum_{j=1}^d (1 - \xi_j \bar{\eta}_j) \left(\sqrt{A_j} v_j(\eta) \right)^* \sqrt{A_j} v_j(\xi),$$

where $q_0(\xi) = \sqrt{A_0}v(\xi)$ is a polynomial map from \mathbb{C}^d into \mathbb{C}^r , $r = |\tilde{\mathcal{I}}|$. The existence of the isometry $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in (23) is guaranteed by Theorem 5.3

part c). Next we explicitly derive one such possible $ABCD$ -representation for f_p . Let

$$g(\xi) = \begin{pmatrix} q_1(\xi) \\ \vdots \\ q_d(\xi) \end{pmatrix}.$$

For $E(\xi)$ in (24), from

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I \\ E(\xi)g(\xi) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} f_p \\ q_0 \end{pmatrix}(\xi) \\ g(\xi) \end{pmatrix}, \quad (25)$$

we immediately get

$$A = \begin{pmatrix} f_p \\ q_0 \end{pmatrix}(0) \in \mathbb{C}^{m+r} \quad \text{and} \quad C = g(0) = \begin{pmatrix} q_1(0) \\ \vdots \\ q_d(0) \end{pmatrix} \in \mathbb{C}^{|\mathcal{I}_1|+\dots+|\mathcal{I}_d|}.$$

Note next that to determine D and B in (25) we need to solve

$$DE(\xi)g(\xi) = g(\xi) - C \quad \text{and} \quad BE(\xi)g(\xi) = \begin{pmatrix} f_p \\ q_0 \end{pmatrix}(\xi) - A. \quad (26)$$

We start by determining the entries of the matrix D , which we write in the block form

$$D = \begin{pmatrix} D_{11} & \dots & D_{1d} \\ \vdots & & \vdots \\ D_{d1} & \dots & D_{dd} \end{pmatrix}, \quad D_{ij} \in \mathbb{C}^{|\mathcal{I}_i| \times |\mathcal{I}_j|}.$$

We index the entries $D_{ij}(\beta, \gamma)$ in the block D_{ij} according to the lexicographical ordering of $\beta \in \mathcal{I}_i$ and $\gamma \in \mathcal{I}_j$. We first observe that, due to $q_j \in \mathbb{C}[\xi_1, \dots, \xi_j]$ (see Remark 4.3) and by the first identity in (26), the matrix D is block lower triangular. Then, from the first identity in (26), for the blocks in the i -th row of D we get

$$\sum_{j=1}^i \xi_j D_{ij} q_j(\xi) = q_i(\xi) - q_i(0), \quad i = 1, \dots, d,$$

where, by (22), the entries of $q_j(\xi)$ are either equal to zero or are scaled monomials $\sqrt{A_j(\beta, \beta)}(\xi_1, \dots, \xi_j)^\beta$, $\beta \in \mathcal{I}_j$. For each $i = 1, \dots, d$, we proceed as follows. Choose a non-negative entry in $q_i(\xi) - q_i(0)$. It corresponds to a non-zero diagonal element $\sqrt{A_i(\beta, \beta)}$ for some $\beta \in \mathcal{I}_i$.

Case 1: If $\beta = (\beta_1, \dots, \beta_i)$ with $\beta_i > 0$, then set $j = i$ and $\gamma = (\beta_1, \dots, \beta_i - 1) \in \mathcal{I}_i$.

Case 2: If $\beta = (\beta_1, \dots, \beta_j, 0, \dots, 0)$ with $j < i$ and $\beta_j > 0$, then set $\gamma = (\beta_1, \dots, \beta_j - 1) \in \mathcal{I}_j$.

By (20), we get that $I(\beta) \subseteq I(\gamma)$, which implies that $A_j(\gamma, \gamma) \geq A_i(\beta, \beta) > 0$. Define

$$D_{ij}(\beta, \gamma) = \sqrt{\frac{A_i(\beta, \beta)}{A_j(\gamma, \gamma)}}.$$

Note that, due to the structure of $q_i(\xi)$, the block D_{ij} has at most one non-negative entry $D_{ij}(\beta, \gamma)$ in each row. Also, for $i = j$ (Case 1) and due to $\gamma_i < \beta_i$, the blocks D_{ii} are lower triangular, with zeros on the main diagonal. This implies that $DE(\xi)$ is nilpotent.

Similarly, we determine the non-zero elements of the matrix B , which we write as a block matrix of the form

$$B = \begin{pmatrix} B_{11} & \dots & B_{1d} \\ B_{21} & \dots & B_{2d} \end{pmatrix}, \quad B_{1j} \in \mathbb{C}^{|G'| \times |\mathcal{I}_j|}, \quad B_{2j} \in \mathbb{C}^{|\tilde{\mathcal{I}}| \times |\mathcal{I}_j|}.$$

Recall that the second identity in (26) is of the form

$$\begin{pmatrix} f_p \\ q_0 \end{pmatrix}(\xi) - A = \begin{pmatrix} \left(m^{1/2} \sum_{\alpha \in \tilde{\mathcal{I}}} p(\alpha_\chi + M\alpha) \xi^\alpha \right)_{\chi \in G'} \\ \sqrt{A_0} v(\xi) \end{pmatrix} - A$$

with $v(\xi) = (\xi^\alpha : \alpha \in \tilde{\mathcal{I}})^T$. By the same argument as above, for each $\alpha \in \tilde{\mathcal{I}}$ we determine $j \in \{1, \dots, d\}$ and $\gamma \in \mathcal{I}_j$ such that $A_j(\gamma, \gamma) > 0$ and $\alpha = (\gamma_1, \dots, \gamma_j + 1, 0, \dots, 0)$. Then non-zero entries of $B_{1,j}$ blocks are defined by

$$B_{1j}(\chi, \gamma) = m^{1/2} \frac{p(\alpha_\chi + M\alpha)}{\sqrt{A_j(\gamma, \gamma)}}, \quad \chi \in G'.$$

Analogously for the blocks B_{2j} , $j = 1, \dots, d$. □

4.2 Matrix factorization: univariate case

We consider the univariate case where $M = m \in \mathbb{N}$ and $m \geq 2$. We use the results of system theory to give an alternative proof of [15, Theorem

4.1] which shows how to construct a tight frame with m generators. With notation in (15), this requires us to find a matrix factorization

$$I_m - f_p(\xi)f_p(\xi)^* = U(\xi)U(\xi)^*, \quad \xi \in \mathbb{T}, \quad (27)$$

where U is a polynomial matrix of dimension $m \times m$. Note that (27) are the UEP identities for (4), written in terms of the polyphase components f_p of p instead of the G -shifts F_p . (Passing from the vector F_p to f_p eliminates the dependencies among the components of F_p .) Then the columns of $U = (u_{\chi,j})_{\chi \in G', j=1,\dots,m}$ define the polyphase components $\tilde{a}_{j,\chi} = u_{\chi,j}$ of each trigonometric polynomial a_j in (4), i.e.

$$a_j(z) = \sum_{\chi \in G'} z^{-\alpha_\chi} u_{\chi,j}(\xi), \quad \xi = z^M \in \mathbb{T}.$$

The following result shows that such a matrix U can be constructed by the scalar Riesz-Fejer lemma and the adjunction formula in Proposition 5.7.

Lemma 4.5. *Assume that $f : \mathbb{D} \rightarrow \mathbb{C}^m$ is a polynomial map with $\|f(\xi)\| \leq 1$ in \mathbb{D}^1 . Then there exist polynomial maps $U : \mathbb{C} \rightarrow \mathbb{C}^{m \times m}$ of degree $n = \deg f$ and $k : \mathbb{C} \rightarrow \mathbb{C}^{m \times n}$ of degree less than n such that*

$$I_m - f(\xi)f(\eta)^* = U(\xi)U(\eta)^* + (1 - \xi\bar{\eta})k(\xi)k(\eta)^*, \quad \xi, \eta \in \mathbb{C}.$$

Proof. Due to $\|f(\xi)\| \leq 1$ in \mathbb{D}^1 and by the Riesz-Fejer Lemma, there is a polynomial $q_0 \in \mathbb{C}[\xi]$ of degree $n = \deg f$, such that

$$\|f(\xi)\|^2 + |q_0(\xi)|^2 \equiv 1, \quad \xi \in \mathbb{T}^1.$$

In other words, the polynomial function $f_1 := \begin{pmatrix} f \\ q_0 \end{pmatrix} : \mathbb{C} \rightarrow \mathbb{C}^{m+1}$ is inner. By Theorem 5.5, there is a polynomial map $q_1 : \mathbb{C} \rightarrow \mathbb{C}^n$ such that

$$1 - f(\eta)^*f(\xi) = q_0(\eta)^*q_0(\xi) + (1 - \xi\bar{\eta})q_1(\eta)^*q_1(\xi), \quad \xi, \eta \in \mathbb{D}^1.$$

Corollary 5.6 implies that the inner function f_1 possesses the representation

$$f_1(\xi) = \begin{pmatrix} A \\ A_0 \end{pmatrix} + \xi \begin{pmatrix} B \\ B_0 \end{pmatrix} (I - \xi D)^{-1} C,$$

where the matrix

$$\begin{pmatrix} A & B \\ A_0 & B_0 \\ C & D \end{pmatrix} : \begin{matrix} \mathbb{C} \\ \oplus \\ \mathbb{C}^n \end{matrix} \longrightarrow \begin{matrix} \mathbb{C}^{m+1} \\ \oplus \\ \mathbb{C}^n \end{matrix}$$

is isometric and $D \in \mathbb{C}^{n \times n}$ is nilpotent.

Using the adjunction formula of Proposition 5.7, we obtain

$$f^*(\xi) = A^* + \xi C^* k^*(\xi),$$

where $k(\xi) = B(I - \xi D)^{-1} \in \mathbb{C}^{m \times n}$ is also a polynomial map of degree less than n . (Note that k is denoted by q_1 in Proposition 5.7.) Since the matrix $\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}$ is contractive, we can choose an extension to an isometry

$$\begin{pmatrix} A^* & C^* \\ X & Y \\ B^* & D^* \end{pmatrix} \quad (28)$$

in the following way: we first extend the co-isometry $\begin{pmatrix} A^* & A_0^* & C^* \\ B^* & B_0^* & D^* \end{pmatrix}$ to a unitary matrix

$$\begin{pmatrix} A^* & A_0^* & C^* \\ X & X_0 & Y \\ B^* & B_0^* & D^* \end{pmatrix}$$

and then drop the middle columns indexed by 0. This shows that the isometric extension (28) exists with $X \in \mathbb{C}^{m \times m}$, $Y \in \mathbb{C}^{m \times n}$. We define the polynomial function $U(\xi) \in \mathbb{C}^{m \times m}$ by

$$U^*(\xi) = X + \xi Y k^*(\xi)$$

and obtain the claim. \square

4.3 Bivariate example: piecewise linear box-spline

The following simple, but educational, example illustrates the result of Corollary 4.4 in the bivariate case, where p is the polynomial associated with the linear three-directional box-spline. In particular, it shows how to derive the $ABCD$ -representation of f_p .

Example 4.6. Let $M = 2I_2$, $m = 4$, and consider

$$p(z_1, z_2) = \frac{1}{8} (1 + z_1 + z_2 + 2z_1 z_2 + z_1 z_2^2 + z_1^2 z_2 + z_1^2 z_2^2).$$

Let $\xi_j = z_j^2$, $j = 1, 2$ and $v(\xi) = (1 \quad \xi_1 \quad \xi_2 \quad \xi_1 \xi_2)^T$. Then

$$f_p(\xi) = \left(m^{1/2} \tilde{p}_\chi(\xi) \right)_{\chi \in G'} = \frac{1}{4} \begin{pmatrix} 1 + \xi_1 \xi_2 \\ 1 + \xi_2 \\ 1 + \xi_1 \\ 2 \end{pmatrix}.$$

Using the construction in the proof of Theorem 4.2, we get

$$1 - f_p(\eta)^* f_p(\xi) = v(\eta)^* A_0 v(\xi) + \sum_{j=1}^2 (1 - \xi_j \bar{\eta}_j) v_j(\eta)^* A_j v_j(\xi)$$

with $v_1(\xi) = 1$, $v_2(\xi) = \begin{pmatrix} 1 & \xi_1 \end{pmatrix}^T$,

$$A_0 = \frac{1}{16} \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \frac{1}{4} \quad \text{and} \quad A_2 = \frac{1}{8} \text{diag}(1, 1).$$

Note that the positive semi-definite matrix A_0 has rank 3 and admits the factorization

$$A_0 = H_0^T H_0 \quad \text{with} \quad H_0 = \frac{1}{4} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

This yields an sos decomposition of length 3 for $1 - f_p(\xi)^* f_p(\xi)$ on \mathbb{T}^2 , namely

$$1 - f_p(\xi)^* f_p(\xi) = q_0(\xi)^* q_0(\xi), \quad q_0(\xi) = H_0 v(\xi) = \frac{1}{4} \begin{pmatrix} 1 - \xi_1 \\ 1 - \xi_2 \\ 1 - \xi_1 \xi_2 \end{pmatrix}.$$

It also allows us to extend the vector-function f_p to an inner function

$$f(\xi) := \begin{pmatrix} f_p \\ q_0 \end{pmatrix}(\xi) = \frac{1}{4} \begin{pmatrix} 1 + \xi_1 \xi_2 \\ 1 + \xi_2 \\ 1 + \xi_1 \\ 2 \\ 1 - \xi_1 \\ 1 - \xi_2 \\ 1 - \xi_1 \xi_2 \end{pmatrix}$$

and have the bilinear representation

$$1 - f_p(\eta)^* f_p(\xi) = q_0(\eta)^* q_0(\xi) + (1 - \bar{\eta}_1 \xi_1) q_1(\eta)^* q_1(\xi) + (1 - \bar{\eta}_2 \xi_2) q_2(\eta)^* q_2(\xi),$$

where

$$q_1(\xi) = \sqrt{A_1} v_1(\xi) = \frac{1}{2}, \quad q_2(\xi) = \sqrt{A_2} v_2(\xi) = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 \\ \xi_1 \end{pmatrix}.$$

Let $g = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$. Next, we use the result of Corollary 4.4, and derive the following $ABCD$ -decomposition of the inner function

$$f(\xi) = A + BE(\xi)(I - DE(\xi))^{-1}C, \quad E(\xi) = \text{diag}(\xi_1, \xi_2, \xi_2),$$

where the block matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{4} \left(\begin{array}{c|ccc} 1 & 0 & 0 & 2\sqrt{2} \\ 1 & 0 & 2\sqrt{2} & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2\sqrt{2} & 0 \\ 1 & 0 & 0 & -2\sqrt{2} \\ \hline 2 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & 0 \end{array} \right)$$

is an isometry (see subsection 5.3 from Appendix for details). The blocks were computed by solving the system

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 \\ E(\xi)g(\xi) \end{pmatrix} = \begin{pmatrix} f(\xi) \\ g(\xi) \end{pmatrix}.$$

Thus, we immediately have $A = f(0)$ and $C = g(0)$. Moreover, $g(\xi) = C + DE(\xi)g(\xi)$ uniquely determines D , and, likewise, $f(\xi) = A + BE(\xi)g(\xi)$ uniquely determines B .

If we apply the construction of [22] for the definition of tight wavelet frames, we will obtain 7 trigonometric polynomials a_1, \dots, a_7 which satisfy the UEP for the given trigonometric polynomial p in Example 4.6. We next show that, by using a shorter extension to an inner function by only 2 additional polynomials, and in combination with the adjunction formula of Proposition 5.7, we reduce the number of trigonometric polynomials to $N = 5$. Moreover, all the corresponding frame generators have small support in $[0, 2]^2$ and every mask has at most 7 nonzero coefficients. Hereby, we improve the existing constructions of tight wavelet frames for the three-directional box-spline B_{111} in [6, 8, 22], where 6 generators with larger support were constructed.

Example 4.7. We let $M = 2I$, $m = 4$, and

$$f_p(\xi) = \left(m^{1/2} \tilde{p}_\chi(\xi) \right)_{\chi \in G'} = \frac{1}{4} \begin{pmatrix} 1 + \xi_1 \xi_2 \\ 1 + \xi_2 \\ 1 + \xi_1 \\ 2 \end{pmatrix}$$

as in Example 4.6.

First, we make use of [22, Example 5.2] and choose another extension of f_p to an inner function by only 2 polynomials (rather than 3 in Example 4.6), namely

$$\tilde{f}(\xi) := \begin{pmatrix} f_p \\ \tilde{q}_0 \end{pmatrix}(\xi) = \frac{1}{4} \begin{pmatrix} 1 + \xi_1 \xi_2 \\ 1 + \xi_2 \\ 1 + \xi_1 \\ 2 \\ \frac{\sqrt{6}}{2}(1 - \xi_1) \\ \frac{\sqrt{2}}{2}(2 - \xi_2 - \xi_1 \xi_2) \end{pmatrix}.$$

Simple computation yields

$$1 - f_p(\eta)^* f_p(\xi) - \tilde{q}_0(\eta)^* \tilde{q}_0(\xi) = \frac{1}{4}(1 - \xi_1 \bar{\eta}_1) + \frac{1}{32}(1 - \xi_2 \bar{\eta}_2)(3 + \xi_1 + \bar{\eta}_1 + 3\xi_1 \bar{\eta}_1).$$

Factorization of the (non-diagonal) semi-definite matrices

$$\tilde{A}_1 = \frac{1}{4} \quad \text{and} \quad \tilde{A}_2 = \frac{1}{32} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

leads to the bilinear representation

$$1 - f_p(\eta)^* f_p(\xi) = \tilde{q}_0(\eta)^* \tilde{q}_0(\xi) + (1 - \bar{\eta}_1 \xi_1) \tilde{q}_1(\eta)^* \tilde{q}_1(\xi) + (1 - \bar{\eta}_2 \xi_2) \tilde{q}_2(\eta)^* \tilde{q}_2(\xi),$$

where

$$\tilde{q}_1(\xi) = \frac{1}{2}, \quad \tilde{q}_2(\xi) = \frac{1}{8} \begin{pmatrix} 2(1 + \xi_1) \\ \sqrt{2}(1 - \xi_1) \end{pmatrix}.$$

The same steps as in Example 4.6 give the following $ABCD$ -decomposition of the inner function

$$\tilde{f}(\xi) = A + BE(\xi)(I - DE(\xi))^{-1}C, \quad E(\xi) = \text{diag}(\xi_1, \xi_2, \xi_2),$$

with isometric block matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{8} \left(\begin{array}{c|ccc} 2 & 0 & 4 & -4\sqrt{2} \\ 2 & 0 & 4 & 4\sqrt{2} \\ 2 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ \hline \sqrt{6} & -2\sqrt{6} & 0 & 0 \\ 2\sqrt{2} & 0 & -4\sqrt{2} & 0 \\ \hline 4 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ \hline \sqrt{2} & -2\sqrt{2} & 0 & 0 \end{array} \right). \quad (29)$$

Next we use the adjunction formula in subsection 5.4 in order to construct the bilinear decomposition

$$I_4 - f_p(\xi)f_p(\eta)^* = u_0(\xi)u_0(\eta)^* + (1 - \bar{\eta}_1\xi_1)u_1(\xi)u_1(\eta)^* + (1 - \bar{\eta}_2\xi_2)u_2(\xi)u_2(\eta)^*. \quad (30)$$

For this purpose, we cut the last two rows of A and B in (29), leaving the contractive block matrix

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ C & D \end{pmatrix} = \frac{1}{8} \left(\begin{array}{c|ccc} 2 & 0 & 4 & -4\sqrt{2} \\ 2 & 0 & 4 & 4\sqrt{2} \\ 2 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ \hline 4 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ \hline \sqrt{2} & -2\sqrt{2} & 0 & 0 \end{array} \right) \quad (31)$$

which represents

$$f_p(\xi) = \tilde{A} + \tilde{B}E(\xi)(I - DE(\xi))^{-1}C. \quad (32)$$

By the adjunction formula, we obtain

$$f_p^*(\xi) = f_p(\bar{\xi})^* = \tilde{A}^* + C^*E(\xi)u^*(\xi), \quad (33)$$

where

$$u(\xi) = \tilde{B}(I - E(\xi)D)^{-1} = \frac{1}{2} \begin{pmatrix} \xi_2 & 1 & -\sqrt{2} \\ 0 & 1 & \sqrt{2} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The polynomial map $u = (u_1, u_2) : \mathbb{C} \rightarrow \mathbb{C}^{4 \times 3}$ defines the functions u_1 (first column) and u_2 (last two columns) in (30). It remains to construct u_0 . The representation (33) refers to the contractive $ABCD$ -matrix

$$\begin{pmatrix} \tilde{A}^* & C^* \\ \tilde{B}^* & D^* \end{pmatrix} = \frac{1}{8} \left(\begin{array}{cccc|ccc} 2 & 2 & 2 & 4 & 4 & 2 & \sqrt{2} \\ 0 & 0 & 4 & 0 & 0 & 4 & -2\sqrt{2} \\ 4 & 4 & 0 & 0 & 0 & 0 & 0 \\ -4\sqrt{2} & 4\sqrt{2} & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

An extension of this matrix to an isometry is obtained by simple linear algebra, adding the following 5 rows

$$(T_0 \quad T_1) = \frac{1}{24} \left(\begin{array}{cccc|ccc} 6\sqrt{3} & 6\sqrt{3} & -2\sqrt{3} & -4\sqrt{3} & -4\sqrt{3} & -2\sqrt{3} & -\sqrt{6} \\ 0 & 0 & -12\sqrt{2} & 0 & 0 & 12\sqrt{2} & 0 \\ 0 & 0 & 0 & 12\sqrt{2} & -12\sqrt{2} & 0 & 0 \\ 0 & 0 & 4\sqrt{6} & -4\sqrt{6} & -4\sqrt{6} & 4\sqrt{6} & 4\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 12\sqrt{3} \end{array} \right).$$

This extension provides the polynomial map $u_0^*(\xi) = T_0 + T_1 E(\xi) u^*(\xi)$, and hence

$$\begin{aligned} u_0(\xi) &= T_0^* + u(\xi) E(\xi) T_1^* \\ &= \frac{1}{12} \begin{pmatrix} \sqrt{3}(3 - \xi_1 \xi_2) & 3\sqrt{2}\xi_2 & -3\sqrt{2}\xi_1 \xi_2 & -\sqrt{6}\xi_1 \xi_2 & -3\sqrt{6}\xi_2 \\ \sqrt{3}(3 - \xi_2) & 3\sqrt{2}\xi_2 & 0 & 2\sqrt{6}\xi_2 & 3\sqrt{6}\xi_2 \\ -\sqrt{3}(1 + \xi_1) & -6\sqrt{2} & -3\sqrt{2}\xi_1 & \sqrt{6}(2 - \xi_1) & 0 \\ -2\sqrt{3} & 0 & 6\sqrt{2} & -2\sqrt{6} & 0 \end{pmatrix}. \end{aligned}$$

Finally, the restriction of (f_p, u_0) to \mathbb{T}^2 defines the matrix $U(\xi)$, $\xi = (z_1^2, z_2^2)$, in the UEP identities

$$I_4 - f_p(\xi) f_p(\xi)^* = U(\xi) U(\xi)^*, \quad \xi \in \mathbb{T}^2.$$

Hence, the number of columns of u_0 and the degree of u_0 determine the number of framelets and their support. We obtain the following 5 trigonometric polynomials

$$\begin{pmatrix} a_1(z) \\ \vdots \\ a_5(z) \end{pmatrix} = \frac{1}{24} \begin{pmatrix} \sqrt{3}(3 + 3z_1 - z_2 - 2z_1 z_2 - z_1^2 z_2 - z_1 z_2^2 - z_1^2 z_2^2) \\ -3\sqrt{2}(2z_2 - z_2^2 - z_1 z_2^2) \\ 3\sqrt{2}(2z_1 z_2 - z_1^2 z_2 - z_1^2 z_2^2) \\ \sqrt{6}(2z_2 - 2z_1 z_2 - z_1^2 z_2 + 2z_1 z_2^2 - z_1^2 z_2^2) \\ -3\sqrt{6}(z_2^2 - z_1 z_2^2) \end{pmatrix}.$$

5 Appendix: Multivariate system analysis

The investigation of the mask p of a tight wavelet frame naturally brings into the picture the class of complex polynomials with a prescribed bound in the polydisk \mathbb{D}^d . Their structure can be better understood from the more general perspective of bounded analytic functions in the polydisk. Fortunately, there is a great deal of accumulated knowledge on this topic, especially arising from a remarkable connection to multivariate system analysis. Without aiming at completeness, the present appendix offers a quick introduction to the subject. The results listed below are used in section 4.

5.1 Single variable

We collect below some classical results which provide the starting point for the more intricate structure of bounded analytic functions in the polydisk.

Let $f(z), |f(z)| \leq 1$, be an analytic function defined in the disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$. Leaving the case of a constant function aside, we can assume that $|f(z)| < 1$ in the disk, and define the function $g(z) = \frac{1+f(z)}{1-f(z)}$, so that $\Re g(z) \geq 0$ for all $|z| < 1$. Let $g_r(z) = g(rz)$, $0 < r < 1$, so that the functions g_r are defined in a neighborhood of the closed disk and $\lim_{r \rightarrow 1} g_r = g$ uniformly on compact subsets of \mathbb{D} . A direct application of Cauchy's formula yields:

$$g_r(w) = \int_{-\pi}^{\pi} \frac{e^{i\theta} + w}{e^{i\theta} - w} \frac{\Re g_r(e^{i\theta}) d\theta}{2\pi} + i\Im g(0).$$

Remark that the measures $d\mu_r = \frac{\Re g_r(e^{i\theta}) d\theta}{2\pi}$ are non-negative, of uniform mass equal to $\Re g(0)$, hence they form a compact set in the weak-* topology of measures on the unit torus. By passing to a limit point we obtain a positive measure μ with the property

$$g(w) = \int_{-\pi}^{\pi} \frac{e^{i\theta} + w}{e^{i\theta} - w} d\mu(\theta) + i\Im g(0). \quad (34)$$

Since the trigonometric polynomials are dense in the space of continuous functions on the torus, we infer that the measure μ is unique with the above property.

Formula (34) is known as the *Riesz-Herglotz representation* of all analytic functions with non-negative real part in the disk. Since \mathbb{D} is simply connected, for any harmonic function $u : \mathbb{D} \rightarrow \mathbb{R}$ there exists an analytic

function $g : \mathbb{D} \rightarrow \mathbb{C}$ such that $u = \Re g$. Putting together these observations we have proved the equivalence between the first two statements in the next theorem.

Theorem 5.1 (Riesz-Herglotz). *Let $g : \mathbb{D} \rightarrow \mathbb{C}$ be an analytic function. The following assertions are equivalent:*

- a). $\Re g \geq 0$;
- b). *There exists a positive measure μ on $\mathbb{T} = \partial\mathbb{D}$, such that (34) holds;*
- c). *The kernel $\frac{g(z) + \overline{g(w)}}{1 - z\overline{w}}$ is positive semi-definite on $\mathbb{D} \times \mathbb{D}$.*

Proof. a) \Rightarrow b) was proved before. If b) holds true, then

$$\frac{g(z) + \overline{g(w)}}{1 - z\overline{w}} = 2 \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{(e^{i\theta} - z)(e^{-i\theta} - \overline{w})},$$

whence c) is true. Finally, c) \Rightarrow a) because a positive semi-definite kernel has non-negative values on the diagonal. \square

It is important to note that *any* positive measure μ on the one-dimensional torus \mathbb{T} can arise in the Riesz-Herglotz parametrization. Also remark that, from $g(z) = \frac{1+f(z)}{1-\overline{f(z)}}$ we infer

$$2 \frac{1 - f(z)\overline{f(w)}}{1 - z\overline{w}} = \frac{1}{1 - f(z)} \frac{g(z) + \overline{g(w)}}{1 - z\overline{w}} \frac{1}{1 - \overline{f(w)}},$$

hence:

Corollary 5.2. *An analytic function f maps the unit disk into itself if and only if the kernel $\frac{1 - f(z)\overline{f(w)}}{1 - z\overline{w}}$ is positive semi-definite, that means*

$$\frac{1 - f(z)\overline{f(w)}}{1 - z\overline{w}} = \sum_{j=1}^N h_j(z)\overline{h_j(w)},$$

where $h_j(z)$ are analytic functions in the disk, and $N \leq \infty$.

Note that, even if $f(z)$ a polynomial, the factors h_j may not be polynomials, or N may be equal to infinity. On one hand, one can factor $f(z) = f_1(z)f_2(z)$ and use induction based on the identity:

$$\frac{1 - f_1(z)f_2(z)\overline{f_1(w)f_2(w)}}{1 - z\overline{w}} = \frac{1 - f_1(z)\overline{f_1(w)}}{1 - z\overline{w}} + f_1(z)\frac{1 - f_2(z)\overline{f_2(w)}}{1 - z\overline{w}}\overline{f_1(w)},$$

having to deal in the end only with a linear factor. But then, even for a constant function $f(z) = c$, the decomposition

$$\frac{1 - |c|^2}{1 - z\bar{w}} = (1 - |c|^2) \sum_{j=0}^{\infty} z^j \bar{w}^j$$

contains infinitely many terms.

The natural framework for finitely determined decompositions of the above type is realized by a class of rational functions, appearing in the celebrated Schur algorithm, see for instance [13].

5.2 Several variables

The analogue of Riesz-Herglotz formula exists in several variables, in general on polyhedral or homogeneous domains. The case of the polydisk was studied by Koranyi and Pukansky [19]. For instance they proved that an analytic function $f(z), z \in \mathbb{D}^d$, is uniformly bounded ($|f(z)| \leq C, z \in \mathbb{D}^d$) if and only if the hermitian kernel

$$\frac{C^2 - f(z)\overline{f(w)}}{\prod_{k=1}^d (1 - z_k \bar{w}_k)}$$

is positive semidefinite. When compared to the single variable case, this formula turns out to be of limited importance for the expected applications. For instance the celebrated Nevanlinna-Pick interpolation theorem does not hold for this positive definite kernel, see [1].

The subtle distinction between 1D and dD with $d \geq 2$ comes from a celebrated result of von Neumann. To be more precise, let $T \in L(H)$ be a linear bounded contraction $\|T\| \leq 1$ acting on a complex Hilbert space. Let $f(z)$ be a strictly contractive analytic function in the disk. A direct consequence of Riesz-Herglotz formula yields

$$I - f(rT)^* f(rT) =$$

$$(I - f(rT)^*)^{-1} \int_{\mathbb{T}} (I - \bar{u}T^*)^{-1} (I - T^*T) (I - uT)^{-1} d\mu(u) (I - f(rT))^{-1} \geq 0,$$

where the positivity is in the sense of Hilbert space operators. By passing to limit with $r \rightarrow 1$ and allowing f to be contractive we obtain *von Neumann's inequality*:

For every analytic function f defined in a neighborhood of the closed unit disk and Hilbert space contractive operator T one has

$$\|f(T)\| \leq \|f\|_{\infty, \mathbb{D}}.$$

Due to an observation of Ando, the above inequality remains true for the bi-disk \mathbb{D}^2 ; but fails for \mathbb{D}^d with $d \geq 3$, see [1]. The contractive analytic functions f in \mathbb{D}^d which satisfy the multi-variate analogue of von-Neumann inequality

$$\|f(T)\| \leq \|f\|_{\infty, \mathbb{D}^d},$$

for every commutative tuple $T = (T_1, \dots, T_d)$ of Hilbert space contractions form the *Schur-Agler class* of functions. Examples of contractive functions in \mathbb{D}^d , $d \geq 3$, which do not belong to this class were known for a long time, see [1]. For instance, the following homogeneous polynomial in three variables

$$g(z_1, z_2, z_3) = z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 \quad (35)$$

satisfies

$$\|g\|_{\infty, \mathbb{D}^3} = 3\sqrt{3},$$

but there exists a commuting triple (T_1, T_2, T_3) of linear contractions acting on a 8 dimensional Hilbert space, so that

$$\|g(T_1, T_2, T_3)\| = 6.$$

For details see [12].

A constructive approach revealing the structure of Schur-Agler functions was completed only during the last decade. The next section collects some results in this direction.

5.3 Multivariate linear systems

The theory of bounded analytic functions in the disk had much to gain from a natural connection with the control theory of linear systems. The resulting interdisciplinary field was vigorously developed during the last forty years, with great benefits for both sides. The multivariate aspects of bounded analytic functions (say in the polydisk) seen as transfer functions of linear systems with multi-time dependence were revealed only during the last decade, see [2] for an excellent survey. We reproduce below a few fundamental facts

of interest for the present work. We deal exclusively with a state-space formulation, with the explicit purpose of parametrizing the polynomials (or analytic functions) we are interested in by structured block-matrices.

The starting point is a quadruple of linear bounded Hilbert space operators $\{A, B, C, D\}$, acting, as a block matrix on two direct sums of Hilbert spaces:

$$\begin{pmatrix} D & C \\ B & A \end{pmatrix} : \begin{matrix} H \\ \oplus \\ X \end{matrix} \longrightarrow \begin{matrix} H \\ \oplus \\ Y \end{matrix}.$$

Moreover, we decompose $H = H_1 \oplus \dots \oplus H_d$ into a direct sum and consider the finite difference scheme

$$\begin{pmatrix} h_1(\alpha + e_1) \\ \vdots \\ h_d(\alpha + e_d) \end{pmatrix} = D \begin{pmatrix} h_1(\alpha) \\ \vdots \\ h_d(\alpha) \end{pmatrix} + Cu(\alpha),$$

$$y(\alpha) = Bh(\alpha) + Au(\alpha), \quad \alpha \in \mathbb{N}^d.$$

Above e_k are the generators of the semigroup \mathbb{N}^d . In linear system theory language, $u(\alpha)$ is the input vector, $h(\alpha)$ is the state space vector and $y(\alpha)$ is the output. All vectors running in the respective Hilbert spaces, with \mathbb{N}^d as a multi-time semigroup. Let $E(z) = z_1 I_{H_1} \oplus \dots \oplus z_d I_{H_d} : H \longrightarrow H$ be regarded as a diagonal operator whose diagonal entries dependent linearly on $z_1, \dots, z_d \in \mathbb{C}$. A great deal of stability analysis of the above finite difference scheme can be read from the associated *transfer function*:

$$F(z) = A + BE(z)(I - DE(z))^{-1}C,$$

first defined for small values of $|z|$, and then analytically continued as far as possible. In case the state space H is finite dimensional D is a finite matrix and hence the transfer function is (vector valued) rational.

The remarkable result which establishes the bridge between contractive analytic functions in the polydisk and linear system theory can be stated as follows, as a combination of an older theorem of Agler (see [1]) and a more recent one due to Ball and Trent [3].

Theorem 5.3 (Agler, Ball, Trent). *Let X, Y be Hilbert spaces and let $f : \mathbb{D}^d \longrightarrow L(X, Y)$ be an analytic function. The following are equivalent.*

a). *For every commutative tuple $T = (T_1, \dots, T_d)$ of linear contractive operators acting on a Hilbert space K , von-Neumann's inequality*

$$\sup_{\epsilon_k < 1} \|f(\epsilon_1 T_1, \dots, \epsilon_d T_d)\| \leq 1,$$

holds;

b). *There exist auxiliary Hilbert spaces H_k and analytic functions $L_k : \mathbb{D}^d \longrightarrow L(X, H_k)$ such that*

$$I - f(w)^* f(z) = \sum_{k=1}^d (1 - \overline{w_k} z_k) L_k(w)^* L_k(z), \quad z, w \in \mathbb{D}^d;$$

c). *There exists an auxiliary Hilbert space $H = H_1 \oplus \dots \oplus H_d$ and a unitary operator*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{matrix} X \\ \oplus \\ H \end{matrix} \longrightarrow \begin{matrix} Y \\ \oplus \\ H \end{matrix},$$

such that

$$f(z) = A + BE(z)(I - DE(z))^{-1}C, \quad z \in \mathbb{D}^d.$$

The meaning of $f(\epsilon_1 T_1, \dots, \epsilon_d T_d) \in L(X, Y) \otimes L(K)$ can be made precise by the Riesz-Dunford functional calculus, or by a formal substitution of z_k by T_k in a power series expansion of the function f . Remember that $E(z) = z_1 I_{H_1} \oplus \dots \oplus z_d I_{H_d} : H \longrightarrow H$ is a diagonal operator, linear in the variables z .

In practice it is sometimes useful to relax condition c) by asking only that the 2×2 block operator is contractive. In this case, denoting $g(z) = (I - DE(z))^{-1}C$, or equivalently $g(z) = C + DE(z)g(z)$, we find

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I \\ E(z)g(z) \end{pmatrix} = \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}.$$

Thus,

$$\|f(z)\|^2 + \|g(z)\|^2 \leq 1 + \|E(z)g(z)\|^2 \leq 1 + \|g(z)\|^2$$

and, therefore, $\|f(z)\| \leq 1$ for all $z \in \mathbb{D}^d$. Since every contractive operator admits a unitary dilation (with the price of increasing the Hilbert space H), all functions f constructed above (from a contractive block operator) belong to Schur-Agler's class.

A constructive approach for determining the A, B, C, D matrices from a function $f(z)$ appears in an early article by Kummert [20]. See also the monograph [4] and the D-module approach to such questions of system theory proposed in [27].

For the functions f that extend to the closed polydisk, we derive the following defect (from unity) formula. Due to $E(z)^* E(z) = I$ for $z \in \mathbb{T}^d$, we

have, for $z \in \mathbb{T}^d$,

$$\begin{aligned} I - f(z)^* f(z) &= \begin{pmatrix} I \\ E(z)g(z) \end{pmatrix}^* \begin{pmatrix} I \\ E(z)g(z) \end{pmatrix} - \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}^* \begin{pmatrix} f(z) \\ g(z) \end{pmatrix} = \\ &= \begin{pmatrix} T \begin{pmatrix} I \\ E(z)g(z) \end{pmatrix} \end{pmatrix}^* \begin{pmatrix} T \begin{pmatrix} I \\ E(z)g(z) \end{pmatrix} \end{pmatrix}, \end{aligned}$$

where

$$I - T^*T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (36)$$

Summing up, we are led to the following result.

Theorem 5.4. *Let $f(z) \in L(X, Y)$ be an operator valued polynomial map belonging to the Schur-Agler class. If a contractive block-matrix realization of $f(z)$ exists and $(I - DE(z))^{-1}C$ is a polynomial, then there are Hilbert spaces Y_0, Y_1, \dots, Y_d and polynomial maps $q_j(z) \in L(X, Y_j)$ with the property*

$$I - f(w)^* f(z) = q_0(w)^* q_0(z) + \sum_{k=1}^d (1 - \overline{w_k} z_k) q_k(w)^* q_k(z), \quad z \in \mathbb{C}^d.$$

Proof. Let us split the matrix T in (36) as $T = (A_0, B_0) : X \oplus H \longrightarrow Y_0$ where Y_0 is an auxiliary Hilbert space of dimension not exceeding the rank of T . Consequently, the matrix

$$V = \begin{pmatrix} A_0 & B_0 \\ A & B \\ C & D \end{pmatrix}$$

is isometric. Hence

$$\begin{pmatrix} q_0 \\ f \end{pmatrix} = \begin{pmatrix} A_0 \\ A \end{pmatrix} + \begin{pmatrix} B_0 \\ B \end{pmatrix} E(z)(I - DE(z))^{-1}C$$

is an operator valued polynomial map. Since V is an isometry, denoting $g(z) = (I - DE(z))^{-1}C$, we find

$$\|q_0(z)\|^2 + \|f(z)\|^2 + \|g(z)\|^2 = 1 + \|E(z)g(z)\|^2$$

and the conclusion follows by polarization. \square

In the case $f(z)$ is an inner polynomial, i.e. $\|f(z)\| = 1$ on \mathbb{T}^d , the second condition in the statement of Theorem 5.4 is automatically satisfied. This fact is stated in the following result proved in [10].

Theorem 5.5 (Cole-Wermer). *Assume that X, Y are finite dimensional Hilbert spaces and $f(z) \in L(X, Y)$ is a polynomial in the Schur-Agler class such that $f(z)^* f(z) = I$ for all $z \in \mathbb{T}^d$. Then*

$$I - f(w)^* f(z) = \sum_{k=1}^d (1 - \overline{w_k} z_k) q_k(w)^* q_k(z),$$

where Y_k are finite dimensional Hilbert spaces, $q_k(z) \in L(X, Y_k)$, $1 \leq k \leq d$, are polynomial maps, and

$$\max(\deg q_1, \dots, \deg q_d) < \deg f.$$

Moreover, the spaces Y_k can be chosen so that $\dim Y_k \leq \dim X \times \dim \mathbb{C}_{\deg f - 1}[z]$.

Proof. Fix $\omega \in \mathbb{T}^d$ and restrict the Agler's decomposition to the ray pointing at ω

$$I - f(r\omega)^* f(r\omega) = (1 - r^2) \sum_{k=1}^d q_k(r\omega)^* q_k(r\omega), \quad r < 1. \quad (37)$$

Since $f(\omega)^* f(\omega) = I$, the quotient

$$\frac{I - f(r\omega)^* f(r\omega)}{1 - r^2} = \frac{1}{1 + r} [f(\omega)^* \frac{f(\omega) - f(r\omega)}{1 - r} + \frac{f(\omega)^* - f(r\omega)^*}{1 - r} f(r\omega)] \quad (38)$$

is a rational function without poles on the positive semi-axis, with polynomial growth at infinity of order $2 \deg p - 2$. The power expansion at zero of the factors q_k is

$$q_k(r\omega) = \sum_{\alpha \in \mathbb{N}^d} q_{k,\alpha} r^{|\alpha|} \omega^\alpha, \quad q_{k,\alpha} : X \rightarrow Y_k,$$

with convergence assured for $0 \leq r < 1$.

Next we free $\omega \in \mathbb{T}^d$ and consider the zero-th order Fourier coefficient of the decomposition (37). The right hand side of (38) is an analytic function in r with

$$\sum_{k=1}^d \sum_{\alpha \in \mathbb{N}^d} q_{k,\alpha}^* q_{k,\alpha} r^{2|\alpha|}$$

convergent for $r < 1$. Moreover, it is rational on the semi-axis $r \in [0, \infty)$ with denominator $(1 + r)$ and with polynomial growth at infinity of order $2 \deg p - 2$. Hence, the analytic function, as the function of r ,

$$(1 + r) \sum_{k=1}^d \sum_{\alpha \in \mathbb{N}^d} q_{k,\alpha}^* q_{k,\alpha} r^{2|\alpha|}$$

is a polynomial. As all its coefficients $q_{k,\alpha}^* q_{k,\alpha}$ are non-negative, we conclude that the operator coefficients $q_{k,\alpha}$ vanish for $|\alpha| \geq \deg p$. This allows us to choose which implies that $\dim Y_k \leq \dim X \cdot \dim \mathbb{C}_{\deg p-1}[z]$. \square

For more details on the above proof and its immediate implications we refer to [10]. The following consequence of Theorem 5.5 is of interest. It states that in the univariate case the matrix D is nilpotent.

Corollary 5.6. *Assume that $f \in L(\mathbb{C}, \mathbb{C}^m)$ is a polynomial and $f(z)^* f(z) = 1$ on \mathbb{T}^1 . Then there exists a realization $f(z) = A + zB(I - zD)^{-1}C$ with an isometry*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{matrix} \mathbb{C} \\ \oplus \\ \mathbb{C}^n \end{matrix} \longrightarrow \begin{matrix} \mathbb{C}^m \\ \oplus \\ \mathbb{C}^n \end{matrix}, \quad n \leq \deg f,$$

and $\det(I - zD)^{-1} = 1$.

Proof. As we deal with the univariate case, we know that f belongs to Schur-Agler class. This implies the existence of a corresponding minimal $ABCD$ -representation. By Theorem 5.5,

$$q_1(z) = (I - zD)^{-1}C = \sum_{\alpha=0}^{\deg f-1} q_1(\alpha)z^\alpha, \quad q_1(\alpha) : \mathbb{C} \rightarrow Y_1,$$

where

$$Y_1 = \text{span}\{q_1(\alpha) : \alpha = 0, \dots, \deg f - 1\} = \mathbb{C}^n, \quad n \leq \deg f.$$

Then the identity $zDq_1(z) = q_1(z) - q_1(0)$ implies

$$\begin{aligned} Dq_1(\alpha) &= q_1(\alpha + 1), \quad \alpha = 0, \dots, n-2, \\ Dq_1(n-1) &= 0. \end{aligned}$$

Therefore, D is nilpotent and $D^n = 0$. \square

5.4 Adjunction formula

The class of Schur-Agler functions in the polydisk is closed under Hilbert space conjugation. The simple adjunction formula below has direct implications to tight wavelet frames, as seen in the body of the present article.

Proposition 5.7. *Let $p(z) \in L(X, Y)$ be a polynomial in the Schur-Agler class of the polydisk \mathbb{D}^d :*

$$p(z) = A + BE(z)(I - DE(z))^{-1}C.$$

Write $p^(\zeta) = p(\bar{\zeta})^*$. Then*

$$p^*(\zeta) = A^* + C^*E(\zeta)(I - D^*E(\zeta))^{-1}B^*.$$

*If, in addition $(I - D^*E(\zeta))^{-1}B^*$ is a polynomial function, then*

$$I - p(z)p(w)^* = q_0(z)q_0(w)^* + \sum_{k=1}^d (1 - z_k \overline{w_k}) q_k(z)q_k(w)^*$$

with matrix valued polynomial functions q_0, \dots, q_d .

Remark 5.8. The assumption that $(I - D^*E(z))^{-1}B^*$ is a polynomial function is satisfied e.g. if $\det(I - D^*E(z)) = 1$. Writing $D^*E(z)$ as a linear pencil

$$D^*E(z) = D^*p_1z_1 + \dots + D^*p_dz_d,$$

with mutually orthogonal projections that add up to the identity $p_1 + p_2 + \dots + p_d = I$, we infer that, for every point $z \in \mathbb{C}^d$, the operator $D^*E(z)$ is nilpotent. Indeed, it suffices to consider the equation

$$\det(I - \zeta D^*E(z)) = 1, \quad \zeta \in \mathbb{C}, \quad z \in \mathbb{C}^d,$$

and put $D^*E(z)$ in the upper-triangular form. By taking adjoints, this amounts to the condition that the linear pencil $E(z)D$ is nilpotent for all $z \in \mathbb{C}^d$. By cyclic invariance

$$\det(I - E(z)D) = \det(I - DE(z))$$

hence the original assumption is equivalent to the fact that the linear pencil $DE(z)$ consists of nilpotent linear transformations.

Greg Knese and collaborators have recently revealed many new details about Agler's decomposition of a positive kernel on the polydisk, see [18].

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